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Realisations of the Representations of para-Fermi Algebra in Fock Space of Bose Operators: Part II

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Abstract

Using the method given in Part I of this series (this volume, pp. 109–114), Green's isomorphisms of the para-Fermi algebra $\mathbf{F}_{2n}^{p}$ with $2n$ generators into $\epsilon_{2n+1}^{p(2)}$ are constructed. All the representations of the para-Fermi algebra are realised in Fock space $\mathscr{H}_{2^n}^{1}$ of 2^n Bose operators.

1. *Introduction*

In Part I of this series (this volume, pp. 109–114), a constructive method was proposed for expressing the generators of the para-Fermi algebra as functions of the generators of the para-Bose algebra and for finding all the irreducible representations of the para-Fermi algebra in the Fock space of Bose operators.

Since no difficulties arise in applying the method for para-Fermi algebras with more than two generators in Part I, for simplicity the case of para-Fermi algebras $\mathbf{F}_2^p$‡ with two generators only was considered. Using the simple matrix representation of the generators of the para-Fermi algebra $\mathbf{F}_2^p$ of arbitrary order of parastatistics p given by Green (1953), isomorphic mappings of $\mathbf{F}_2^p$ into $\epsilon_{p+2}^{(2)}$ were constructed.§ In the Fock space $\mathscr{H}_2^1$ of two Bose operators all the irreducible representations of the para-Fermi algebra have been found; every subspace H_n, $n = 1, 2, \ldots$ of $\mathscr{H}_2^1$ spanned on the n-particle states of Bose operators is transformed irreducibly under the transformations induced by the para-Fermi algebra with parastatistics n. A consistent interpretation was given of the vectors of this subspace as states of para-Fermi operators with parastatistics n. So the Fock space $\mathscr{H}_2^1$

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‡ For $\mathbf{F}_2^p$ in Part I the notation $\mathbf{F}^p$ was used.

§ See Part I, equation (3.1), p. 111.

was decomposed into a direct sum of spaces H_n spanned on the states of para-Fermi operators with parastatistics $n = 1, 2, \ldots$.

Since, for physical application, systems of many Fermi particles in different states are needed, in the present paper, using the method proposed in Part I, we construct the explicit formulas of the isomorphisms of $\mathbf{F}^{1}_{2n}$ (the Fermi algebra with $2n$ generators) in the form of equation (3.1) of Part I† and we find all the representations of the para-Fermi algebra with $2n$ generators in the Fock space $\mathscr{H}^{1}_{2^n}$ of 2^n Bose operators.

In Section 2 of the present paper we find a simple matrix representation of the algebra $\mathbf{F}^{1}_{2n}$, and making use of it we construct the desired isomorphisms. The representations of $\mathbf{F}^{1}_{2n}$ in the space $\mathscr{H}^{1}_{2^n}$ are found in Section 3.

Everywhere in this paper we shall follow the notations introduced in Part I, except when we have explicitly stated otherwise.

2. *Matrix Representation of the Algebra* $\mathbf{F}^{1}_{2n}$ *and Isomorphic Mapping of* $\mathbf{F}^{1}_{2n}$ *into* $\epsilon^{q(2)}_{2^n+1}$

Let us introduce the following $2^n \otimes 2^n$ matrices:

$$
\begin{aligned}
(\overset{+}{F}_i)_{rs} &= (\overset{+}{F}_i)_{\sum_{k=1}^{n} \nu_k 2^{k-1}+1,\ \sum_{k=1}^{n} \mu_k 2^{k-1}+1} \\
&= (-1)^{\sum_{k=i+1}^{n} \mu_k} \theta(1-\mu_i)\, \delta_{\nu_1 \mu_1} \cdots \delta_{\nu_i \mu_i+1} \cdots \delta_{\nu_n \mu_n} \qquad (2.1)\\
F_i &= (\overset{+}{F}_i)^T
\end{aligned}
$$

where $i = 1, 2, \ldots, n$; and the indices $r, s = 1, 2, 3, \ldots, 2^n$ are written in binary arithmetics, $\mu_k, \nu_k = 0, 1$ for $k = 1, \ldots, n$; $\theta(x) = 0$ for $x \leqslant 0$, 1 for $x > 0$; $\delta_{\mu\nu}$ being the Kronecker symbol. Then the following proposition is correct:

Proposition

The set (2.1) of $2n$ matrices forms a Fermi algebra.

Proof: Indeed the commutation relations of the matrices (2.1) are those of a Fermi algebra. For definiteness, let $j > i$.

$$
\begin{aligned}
([\overset{+}{F}_i, \overset{+}{F}_j]_+)_{rs} &= (\overset{+}{F}_i)_{rq}(\overset{+}{F}_j)_{qs} + (\overset{+}{F}_j)_{rq}(\overset{+}{F}_i)_{qs} \\
&= (-1)^{\sum_{k=i+1}^{n} \kappa_k + \sum_{k=j+1}^{n} \mu_k} \theta(1-\kappa_i)\theta(1-\mu_j) \\
&\quad .\delta_{\nu_1 \kappa_1} \cdots \delta_{\nu_i \kappa_i+1} \cdots \delta_{\nu_n \kappa_n} . \delta_{\kappa_1 \mu_1} \cdots \delta_{\kappa_j \mu_j+1} \cdots \delta_{\kappa_n \mu_n} \\
&\quad + (-1)^{\sum_{k=j+1}^{n} \kappa_k + \sum_{k=i+1}^{n} \mu_k} \theta(1-\kappa_j)\theta(1-\mu_i) \\
&\quad .\delta_{\nu_1 \kappa_1} \cdots \delta_{\nu_j \kappa_j+1} \cdots \delta_{\nu_n \kappa_n} \delta_{\kappa_1 \mu_1} \cdots \delta_{\kappa_i \mu_i+1} \cdots \delta_{\kappa_n \mu_n} \\
&= \Big((-1)^{\sum_{k=i+1}^{n} \mu_k + \sum_{k=j+1}^{n} \mu_k + 1} + (-1)^{\sum_{k=j+1}^{n} \mu_k + \sum_{k=i+1}^{n} \mu_k}\Big)\theta(1-\mu_i)\theta(1-\mu_j) \\
&\quad .\delta_{\nu_1 \mu_1} \cdots \delta_{\nu_i \mu_i+1} \cdots \delta_{\nu_j \mu_j+1} \cdots \delta_{\nu_n \mu_n} \\
&= 0
\end{aligned}
$$

† See p. 111.

□ All the other commutation relations are checked in the same way.

From the theorem in Part I (p. 111) it follows that the mappings

$$l_q:\overset{+}{\mathscr{F}}_l=\sum_{\mu_1,\ldots,\mu_n=0}^{1}(-1)^{\sum_{k=l+1}^{n}\mu_k}\theta(1-\mu_l)\tfrac{1}{2}\left[\overset{+}{b}{}^{q}_{\sum_{k=1}^{n}\mu_k 2^{k-1}+2^{l-1}+1},\, b^{q}_{\sum_{k=1}^{n}\mu_k 2^{k-1}+1}\right]_+$$

$$\mathscr{F}_l=(\overset{+}{\mathscr{F}}_l)^+ \tag{2.2}$$

of $\mathbf{F}^1_{2n}$ into $\epsilon^{q(2)}_{2^n+1}$ are Green isomorphisms.

For the case $q=1$, $\overset{+}{b}_j$, b_j being Bose operators, these isomorphisms are reduced to isomorphisms of $\mathbf{F}^1_{2n}$ into $\epsilon^{1(2)}_{2^n+1}$,

$$l_1:\overset{+}{\mathscr{F}}_l=\sum_{\mu_1,\ldots,\mu_n=0}^{1}(-1)^{\sum_{k=l+1}^{n}\mu_k}\theta(1-\mu_l)\overset{+}{b}_{\sum_{k=1}^{n}\mu_k 2^{k-1}+2^{l-1}+1}\, b_{\sum_{k=1}^{n}\mu_k 2^{k-1}+1} \tag{2.3}$$

$$\mathscr{F}_l=(\overset{+}{\mathscr{F}}_l)^+$$

So the operators $\overset{+}{\mathscr{F}}_l$, $\mathscr{F}_l$, obtained by $\overset{+}{F}_l$, F_l through the Green isomorphisms, form para-Fermi algebra.

3. *Realisation of the Representations of the para-Fermi Algebra* $\mathbf{F}^1_{2n}$ *into the Fock Space* $\mathscr{H}^1_{2^n}$ *of the para-Bose Algebra* $\mathbf{B}^1_{2^n+1}$

Now we shall show that using the prescription given in Part I we can find all the representations of the para-Fermi algebra (2.3) in the Fock space of 2^n Bose operators.

The basis of the Fock space, as in the previous case, is given in the form

$$|\alpha_1,\ldots,\alpha_{2^n}\rangle=\prod_{j=1}^{2^n}\frac{(\overset{+}{b}_j)^{\alpha_j}}{\sqrt{(\alpha_j!)}}|0\rangle \tag{3.1}$$

In order to write the basis (3.1) in binary arithmetics we must substitute everywhere the index j by the corresponding expression from these arithmetics. Then the action in $\mathscr{H}^1_{2^n}$ induced by the operators (2.3) is the following:

$$\overset{+}{\mathscr{F}}_l|\alpha_1,\ldots,\alpha_{2^n}\rangle=\sum_{\mu_1,\ldots,\mu_n=0}^{1}(-1)^{\sum_{k=l+1}^{n}\mu_k}\theta(1-\mu_l)\cdot$$

$$\cdot\left[\alpha_{\sum_{k=1}^{n}\mu_k 2^{k-1}+1}\cdot\left(\alpha_{\sum_{k=1}^{n}\mu_k 2^{k-1}+2^{l-1}+1}+1\right)\right]^{1/2} \tag{3.2}$$

$$|\alpha_1,\ldots,\alpha_{\sum_{k=1}^{n}\mu_k 2^{k-1}+1}-1,\ldots,\alpha_{\sum_{k=1}^{n}\mu_k 2^{k-1}+2^{l-1}+1}+1,\ldots,\alpha_n\rangle$$

and an analogous expression for the action of the operator $\mathscr{F}_l$.

Then from (3.2) it follows that

$$\mathscr{F}_i(\overset{+}{b}_1)^m|0\rangle=0,\qquad i=1,2,\ldots,n \tag{3.3}$$

So the vector $(\overset{+}{b}_1)^m|0\rangle$ can be considered as a vacuum for the operators $\mathscr{F}_i$, $i = 1, \ldots, n$ in the subspace H_m, $m = 1, 2, \ldots$ of m-particle states of 2^n Bose operators. This is consistent with the result obtained after applying the number-particle operators† on this state.

Moreover, from (3.2) it follows that

$$\overset{+}{\mathscr{F}}_i \overset{+}{\mathscr{F}}_i (\overset{+}{b}_1)^m|0\rangle = m(\overset{+}{b}_1)^m|0\rangle \tag{3.4}$$

So in the space spanned on the vectors

$$|m\rangle = \prod_{i=1}^{n} (\overset{+}{\mathscr{F}}_i)^{\beta_i}(\overset{+}{b}_1)^m|0\rangle \tag{3.5}$$

(β_i being arbitrary non-negative integer) the transformations (3.2) form a para-Fermi algebra with parastatistics m.

In this way, using the method proposed in Part I, we derived all the representations of the para-Fermi algebra with $2n$ operators in the space of 2^n Bose operators.

4. *Discussion*

The foregoing mathematical procedure which follows from Part I enabled us to find the representations of para-Fermi algebra in the Fock space of Bose operators and has the following consequence. A possibility is given for reformulating physical theories in equivalent ones without fermions (and para-fermions). The old and the reformulated theories would be physically indistinguishable.

Future papers will be devoted to further discussions of the problem.

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† See Part I, equation (2.6), p. 110.